EXERCITATIONES ANALYTICAE.

Auctore

L. EVLERO.

I.

Non parum notatu digna videtur relatio, quam inter ſummas harum ſerierum diuergentium

$$1 - 2^m + 3^m - 4^m + 5^m - \text{etc.}$$

et iſtarum conuergentium

$$1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \text{etc.}$$

intercedere olim obſeruaui, et quae ita ſe habet:

$$1 - 2^0 + 3^0 - 4^0 + \text{etc.} = \tfrac{1}{2}$$

$$1 - 2^1 + 3^1 - 4^1 + \text{etc.} = \tfrac{1}{4} = + \tfrac{2\cdot1}{\pi^2}\left(1 + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \text{etc.}\right)$$

$$1 - 2^2 + 3^2 - 4^2 + \text{etc.} = \tfrac{0}{8}$$

$$1 - 2^3 + 3^3 - 4^3 + \text{etc.} = -\tfrac{2}{16} = -\tfrac{2\cdot1\cdot2\cdot3}{\pi^4}\left(1 + \tfrac{1}{3^4} + \tfrac{1}{5^4} + \tfrac{1}{7^4} + \text{etc.}\right)$$

$$1 - 2^4 + 3^4 - 4^4 + \text{etc.} = \tfrac{0}{32}$$

$$1 - 2^5 + 3^5 - 4^5 + \text{etc.} = \tfrac{16}{64} = + \tfrac{2\cdot1\cdot2\cdot3\cdot4\cdot5}{\pi^6}\left(1 + \tfrac{1}{3^6} + \tfrac{1}{5^6} + \tfrac{1}{7^6} + \text{etc.}\right)$$

$$1 - 2^6 + 3^6 - 4^6 + \text{etc.} = \tfrac{0}{128}$$

$$1 - 2^7 + 3^7 - 4^7 + \text{etc.} = -\tfrac{272}{256} = -\tfrac{2\cdot1\cdot2\cdots7}{\pi^8}\left(1 + \tfrac{1}{3^8} + \tfrac{1}{5^8} + \tfrac{1}{7^8} + \text{etc.}\right)$$

$$1 - 2^8 + 3^8 - 4^8 + \text{etc.} = \tfrac{0}{512}$$

$$1 - 2^9 + 3^9 - 4^9 + \text{etc.} = \tfrac{7936}{1024} = + \tfrac{2\cdot1\cdot2\cdots9}{\pi^{10}}\left(1 + \tfrac{1}{3^{10}} + \tfrac{1}{5^{10}} + \tfrac{1}{7^{10}} + \text{etc.}\right)$$

vbi π denotat peripheriam circuli, cuius diameter $= 1$.

2. Hinc concludere licet, in genere inter has series infinitas

$$1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + \text{etc. et } 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \text{etc.}$$

huiusmodi relationem locum habere, vt sit

$$1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + \text{etc.}$$

$$= \frac{2.1.2.3\ldots(n-1)}{\pi^n} N\left(1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \text{etc.}\right)$$

vbi quidem nouimus, quoties n fuerit numerus impar excepto casu $n = 1$, fore $N = 0$; quoties autem sit n numerus par, esse N vel $+1$ vel -1. Scilicet si sit n numerus impariter par formae $4m + 2$, erit $N = +1$, sin autem n numerus pariter par formae $4m$, erit $N = -1$. Vnde cuiusmodi functio N sit ipsius n haud difficulter coniicere licebit; cum

si $n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$ etc.
sit $N = +1, 0, -1, 0, +1, 0, -1, 0, +1, 0, -1, 0$ etc.

3. Neque etiam, si rem attentius perpendamus, casus $n = 1$ huic legi aduersatur, qua fieri debet $N = 0$; nihil enim impedit, quo minus hanc aequalitatem admittamus:

$$1 - 2^0 + 3^0 - 4^0 + \text{etc.} = \tfrac{2}{\pi} 0 \left(1 + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{9} + \text{etc.}\right)$$

quandoquidem seriei

$$1 + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \text{etc.}$$

summa

summa est infinita, vnde vtique fieri potest $\frac{2}{\pi}.0\infty = 1$ seu summae seriei

$$1 - 1 + 1 - 1 + \text{etc.}$$

Quamobrem sine vlla exceptione

si fuerit $n = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9$ etc.
erit $N = 0, +1, 0, -1, 0, +1, 0, -1, 0$ etc.

cui quidem legi innumerae formulae pro N assumendae satisfacere possunt. Verum dubitare non licet, quin simplicissima maximeque naturalis hic locum inueniat, quae est $N = \operatorname{cos.} \frac{n-1}{2}\pi$, denotante hic π angulum duobus rectis aequalem, quoniam sinus totus $= 1$ assumitur, vt π sit semicircumferentia circuli.

4. Admissa ergo hac coniectura, habebimus in genere:

$$1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + \text{etc} = 2\operatorname{cos.} \frac{n-2}{2}\pi . \frac{1.2.3\ldots(n-1)}{\pi^n}$$
$$\left(1 + \frac{1}{3^n} + \frac{1}{5^n} + \text{etc.}\right)$$

siue conuertendo:

$$1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \text{etc.} = \frac{1}{2\operatorname{cos.}\frac{n-2}{2}\pi} . \frac{\pi^n}{1.2.3\ldots(n-1)}$$
$$(1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + \text{etc.})$$

atque ex praecedentibus manifestum est, hanc aequalitatem reuera consistere quoties fuerit n numerus par, neque etiam a veritate discedere casibus, quibus n est numerus impar. Quare si ea vera sit pro casi-

casibus, quibus n est numerus fractus, formulae $1 \cdot 2 \cdot 3 \ldots\ldots (n-1)$ valores per interpolationem assignari debent, qui quidem pro semissibus ita se habent:

$$\text{si } n-1 = \tfrac{1}{2}, \quad \tfrac{3}{2}, \quad \tfrac{5}{2}, \quad \tfrac{7}{2}, \quad \tfrac{9}{2}, \quad \text{etc.}$$

$$\text{sunt } \tfrac{1}{2}\sqrt{\pi};\ \tfrac{1\cdot 3}{2\cdot 2}\sqrt{\pi};\ \tfrac{1\cdot 3\cdot 5}{2\cdot 2\cdot 2}\sqrt{\pi};\ \tfrac{1\cdot 3\cdot 5\cdot 7}{2\cdot 2\cdot 2\cdot 2}\sqrt{\pi};\ \tfrac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 2\cdot 2\cdot 2\cdot 2}\sqrt{\pi}\ \text{etc.}$$

$$\text{et } \cos\tfrac{(n-2)\pi}{2} = \tfrac{1}{\sqrt{2}};\ \tfrac{1}{\sqrt{2}};\ -\tfrac{1}{\sqrt{2}};\ -\tfrac{1}{\sqrt{2}};\ +\tfrac{1}{\sqrt{2}}.$$

5. Pro his ergo casibus habebimus:

$$1+\frac{1}{3\sqrt{3}}+\frac{1}{5\sqrt{5}}+\frac{1}{7\sqrt{7}}+\text{etc.} = +\frac{\sqrt{2}}{1}\,\pi\,(1-\sqrt{2}+\sqrt{3}-\sqrt{4}-\sqrt{5}-\text{etc.})$$

$$1+\frac{1}{3^2\sqrt{3}}+\frac{1}{5^2\sqrt{5}}+\frac{1}{7^2\sqrt{7}}+\text{etc.} = +\frac{2\sqrt{2}}{1\cdot 3}\,\pi^2(1-2\sqrt{2}+3\sqrt{3}-4\sqrt{4}+\text{etc.})$$

$$1+\frac{1}{3^3\sqrt{3}}+\frac{1}{5^3\sqrt{5}}+\frac{1}{7^3\sqrt{7}}+\text{etc.} = -\frac{4\sqrt{2}}{1\cdot 3\cdot 5}\,\pi^3(1-2^2\sqrt{2}+3^2\sqrt{3}-4^2\sqrt{4}+\text{etc.})$$

$$1+\frac{1}{3^4\sqrt{3}}+\frac{1}{5^4\sqrt{5}}+\frac{1}{7^4\sqrt{7}}+\text{etc.} = -\frac{8\sqrt{2}}{1\cdot 3\cdot 5\cdot 7}\,\pi^4(1-2^3\sqrt{2}+3^3\sqrt{3}-4^3\sqrt{4}+\text{etc.})$$

$$1+\frac{1}{3^5\sqrt{3}}+\frac{1}{5^5\sqrt{5}}+\frac{1}{7^5\sqrt{7}}+\text{etc.} = +\frac{16\sqrt{2}}{1\cdot 3\cdot 5\cdot 7\cdot 9}\,\pi^5(1-2^4\sqrt{2}+3^4\sqrt{3}-4^4\sqrt{4}+\text{etc.})$$

etc.

quae aequalitates, an absolute sint verae, pertinaciter asseuerare non ausim; scrutari ergo conuenit, num seriebus per approximationem summatis satisfiat; ac pro prima quidem colligimus

$$1-\sqrt{2}+\sqrt{3}-\sqrt{4}+\sqrt{5}-\text{etc.} = 0,380317 \text{ proxime}$$

qui numerus per $\pi\sqrt{2}$ multiplicatur dat $1,689665$, cui summa seriei

$$1+\frac{1}{3\sqrt{3}}+\frac{1}{5\sqrt{5}}+\frac{1}{7\sqrt{7}}\ \text{etc.}$$

proxime aequalis deprehenditur.

6. Quoniam autem numeris imparibus pro n accipiendis hinc nihil concludi posse videtur, propterea

rea quod alterum membrum nostrae aequationis abit in $\frac{0}{0}$; vt hos valores inuestigemus, pro n ponamus numerum infinite parum excedentem numerum integrum, seu scribamus $n+\omega$ loco n, denotante ω fractionem infinite paruam; atque habebimus:

$$1+\frac{1}{3^{n+\omega}}+\frac{1}{5^{n+\omega}}+\frac{1}{7^{n+\omega}}+\text{etc.}=\frac{1}{2\cos.\frac{n-2+\omega}{n}\pi}\cdot\frac{\pi^{n+\omega}}{1.2\ldots(n-1+\omega)}$$
$$(1-2^{n-1+\omega}+3^{n-1+\omega}-4^{n-1+\omega}+\text{etc.})$$

Hic igitur primo obseruo esse

$$\frac{1}{a^{n+\omega}}=a^{-n-\omega}=a^{-n}(1-\omega\, l\, a)$$

vbi logarithmi naturales seu hyperbolici sunt intelligendi, ita vt sit

$$\frac{1}{a^{n+\omega}}=\frac{1}{a^n}-\frac{\omega\, l\, a}{a^n}.$$

Simili modo erit

$$a^{n-1+\omega}=a^{n-1}+a^{n-1}\omega\, l\, a, \text{ et } \pi^{n+\omega}=\pi^n(1+\omega\, l\,\pi):$$

tum vero fit

$$\cos.\frac{n-2+\omega}{2}\pi=\cos.\frac{n-2}{2}\pi-\tfrac{1}{2}\omega\,\pi\,\sin.\frac{n-2}{2}\pi.$$

Denique cum ostenderim olim formulae $1.2\ldots(n-1+\omega)$ valorem casu $n=1$ esse $=1-0,5772156\,\omega$, si scribamus breuitatis gratia $\lambda=0,5772156649015325$

sumendo $n=1$; 2; 3; 4; 5 etc.

fit $1.2\ldots(n-1+\omega)=1-\lambda\omega$; $1+(1-\lambda)\omega$; $2+(3-2\lambda)\omega$; $6+(11-6\lambda)\omega$; $24+(50-24\lambda)\omega$ etc.

7. Consideremus hinc potissimum casum $n=3$, quandoquidem haec series

$$1+\tfrac{1}{3^3}+\tfrac{1}{5^3}+\tfrac{1}{7^3}+\text{etc.}$$

ita eſt comparata, vt omnes adhuc labores ad eius ſummam inueſtigandam fruſtra ſuſcepti ſint. Cum igitur ſit coſ. $\frac{n-2}{2}\pi = 0$ et ſin. $\frac{n-2}{2}\pi = 1$, noſtra aequatio hanc induet formam:

$$\left.\begin{matrix} 1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\text{etc} \\ -\omega(l1+\frac{l3}{3^3}+\frac{l5}{5^3}+\frac{l7}{7^3}+\text{etc.} \end{matrix}\right\} = \frac{-1}{\pi\,\omega}\,\frac{\pi^3(1+\omega\, l\pi)}{2+(3-2\lambda)\omega}\left\{\begin{matrix} 1-2^2+3^2-4^2+\text{etc.} \\ -\omega(2^2l2-3^2l3+4^2l4-\text{etc.}) \end{matrix}\right\}$$

Verum quia

$$1-2^2+3^2-4^2+\text{etc.}=0,$$

poſito $\omega=0$, habebimus

$$1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\text{etc.}=\tfrac{1}{2}\pi^2(2^2l2-3^2l3+4^2l4-5^2l5+\text{etc.})$$

ſicque ad ſcopum pertingeremus, ſi huius ſeriei logarithmicae

$$2^2l2-3^2l3+4^2l4-5^2l5+\text{etc.}$$

ſummam aſſignare liceret. Simili autem modo pro reliquis poteſtatibus reperitur

$$1+\frac{1}{3^5}+\frac{1}{5^5}+\frac{1}{7^5}+\text{etc.}=\frac{-\pi^4}{1.2.3.4}(2^4l2-3^4l3+4^4l4-5^4l5+\text{etc.})$$

$$1+\frac{1}{3^7}+\frac{1}{5^7}+\frac{1}{7^7}+\text{etc.}=\frac{+\pi^6}{1.2\ldots6}(2^6l2-3^6l3+4^6l4-5^6l5+\text{etc.})$$

$$1+\frac{1}{3^9}+\frac{1}{5^9}+\frac{1}{7^9}+\text{etc.}=\frac{-\pi^8}{1.2\ldots8}(2^8l2-3^8l3+4^8l4-5^8l5+\text{etc.})$$

8. Propoſita ergo nobis ſit haec ſeries infinita:

$$2^2l2-3^2l3+4^2l4-5^2l5+6^2l6-7^2l7+\text{etc.}=Z$$

vt fiat $1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\text{etc.}=\frac{1}{2}\pi\pi Z$,

et quo minus de eius ſumma deſperemus, notetur eſſe

$$l2-l3+l4-l5+l6-\text{etc.}=\tfrac{1}{2}l\tfrac{\pi}{2}.$$

Illa

Illa autem ſeries Z in plures formas transmutari poteſt, veluti

$$Z = l2 - 3l\tfrac{3}{2} + 6l\tfrac{4}{3} - 10l\tfrac{5}{4} + 15l\tfrac{6}{5} - 21l\tfrac{7}{6} + \text{etc. et}$$

$$Z = l\tfrac{2.2}{1.3} + 4l\tfrac{4.4}{3.5} + 9l\tfrac{6.6}{5.7} + 16l\tfrac{8.8}{7.9} + 25l\tfrac{10.10}{9.11} \text{ etc.}$$

$$- 2l\tfrac{3.3}{2.4} - 6l\tfrac{5.5}{4.6} - 12l\tfrac{7.7}{6.8} - 20l\tfrac{9.9}{8.10} - \text{etc.}$$

Si enim in genere ponamus

$$Z = \alpha l\tfrac{2.2}{1.3} - \beta l\tfrac{3.3}{2.4} + \gamma l\tfrac{4.4}{3.5} - \delta l\tfrac{5.5}{4.6} + \varepsilon l\tfrac{6.6}{5.7} - \zeta l\tfrac{7.7}{6.8} \text{ etc.}$$

eſſe debet $2\alpha + \beta = 4$ hincque $\beta = 4 - 2\alpha$ | $\zeta = 18 - 6\alpha$

$\alpha + 2\beta + \gamma = 9$ $\quad\gamma = 1 + 3\alpha$ | $\eta = 9 + 7\alpha$

$\beta + 2\gamma + \delta = 16$ $\quad\delta = 10 - 4\alpha$ | $\theta = 28 - 8\alpha$

$\gamma + 2\delta + \varepsilon = 25$ $\quad\varepsilon = 4 + 5\alpha$ | $\varepsilon = 15 + 9\alpha$

hic vero ſumſimus $\alpha = 1$, vt progreſſio maxime fiat regularis.

9. Haec poſterior forma maxime ad inſtitutum noſtrum videtur accommodata, quoniam logarithmi in ſeries conuergentes reſoluuntur. Hunc in finem pro terminis poſitiuis hac reſolutione vtar; cum quilibet hac forma contineatur

$$xx\,l\frac{4xx}{4xx-1} = -xx\,l\left(1 - \frac{1}{4xx}\right)$$

inde naſcitur haec ſeries infinita:

$$xx\left(\frac{1}{4xx} + \frac{1}{2.2^4x^4} + \frac{1}{3.2^6x^6} + \frac{1}{4.2^8x^8} + \text{etc.}\right) \text{ ſeu haec}$$

$$\frac{1}{2^2} + \frac{1}{2.2^4}.\frac{1}{xx} + \frac{1}{3.2^6}.\frac{1}{x^4} + \frac{1}{4.2^8}.\frac{1}{x^6} + \frac{7}{5.2^{10}}.\frac{1}{x^8} + \text{etc.}$$

Pro terminis autem negatiuis forma generalis eſt

$$-x(x+1)\,l\frac{(2x+1)^2}{4x(x+1)} = -x(x+1)\,l\left(1 + \frac{1}{4x(x+1)}\right),$$

 quae

quae resoluitur in hanc seriem:

$$-\frac{1}{2^2}+\frac{1}{2\cdot 2^4}\cdot\frac{1}{x(x+1)}-\frac{1}{3\cdot 2^6}\cdot\frac{1}{x^2(x+1)^2}-\frac{1}{4\cdot 2^8}\cdot\frac{1}{x^3(x+1)^3}+\text{etc.}$$

vnde valor ipsius Z in has series transformatur:

$$Z=\frac{1}{2^2}(1-1+1-1+\text{etc.})+\frac{1}{2\cdot 2^4}\left(1+\frac{1}{1\cdot 2}+\frac{1}{2^2}+\frac{1}{2\cdot 3}+\frac{1}{3^2}+\frac{1}{3\cdot 4}\ \text{etc.}\right)$$
$$+\frac{1}{3\cdot 2^6}\left(1-\frac{1}{1^2\cdot 2^2}+\frac{1}{2^4}-\frac{1}{2^2\cdot 3^2}+\text{etc.}\right)+\frac{1}{4\cdot 2^8}\left(1+\frac{1}{1^3\cdot 2^3}+\frac{1}{2^6}+\frac{1}{2^3\cdot 3^3}+\frac{1}{3^6}+\text{etc.}\right)$$
$$+\frac{1}{5\cdot 2^{10}}\left(1-\frac{1}{1^4\cdot 2^4}+\frac{1}{2^8}-\frac{1}{2^4\cdot 3^4}+\text{etc.}\right)+\frac{1}{6\cdot 2^{12}}\left(1+\frac{1}{1^5\cdot 2^5}+\frac{1}{2^{10}}+\frac{1}{2^5\cdot 3^5}+\frac{1}{3^{10}}+\text{etc.}\right)$$

10. Quodsi nunc breuitatis gratia ponamus:

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc.}=\alpha\pi^2;$$
$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.}=\beta\pi^4;$$
$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc.}=\gamma\pi^6$$
$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\text{etc.}=\delta\pi^8$$

vbi quidem numeri α, β, γ, δ etc. sunt cogniti; et quia

$$1-1+1-1\ \text{etc.}=\frac{1}{2}\ \text{erit}$$

$$Z=\frac{1}{2^2}\cdot\frac{1}{2}+\frac{1}{2\cdot 2^4}\left(\alpha\pi^2+\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\text{etc.}\right)$$
$$+\frac{1}{3\cdot 2^6}\left(\beta\pi^4-\frac{1}{2^2}-\frac{1}{6^2}-\frac{1}{12^2}-\text{etc.}\right)+\frac{1}{4\cdot 2^8}\left(\gamma\pi^6+\frac{1}{2^3}+\frac{1}{6^3}+\frac{1}{12^3}+\text{etc.}\right)$$
$$+\frac{1}{5\cdot 2^{10}}\left(\delta\pi^8-\frac{1}{2^4}-\frac{1}{6^4}-\frac{1}{12^4}-\text{etc.}\right)+\frac{1}{6\cdot 2^{12}}\left(\varepsilon\pi^{10}+\frac{1}{2^5}+\frac{1}{6^5}+\frac{1}{12^5}\ \text{etc.}\right)$$

vbi iam totum negotium ad summationem harum serierum

$$\frac{1}{2^n}+\frac{1}{6^n}+\frac{1}{12^n}+\frac{1}{20^n}+\text{etc.}$$

est reductum, quarum potestatum radices 2, 6, 12, 20 etc. sunt numeri pronici.

11.

11. Huius autem ſeriei ſinguli termini, quorum forma eſt $\frac{1}{x^n(x+1)^n}$ in partes ſimplicium poteſtatem reſolui poſſunt, quae ita ſe habent:

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

$$\frac{1}{x^2(x+1)^2} = \frac{1}{x^2} + \frac{1}{(x+1)^2} - 2\left(\frac{1}{x} - \frac{1}{x+1}\right)$$

$$\frac{1}{x^3(x+1)^3} = \frac{1}{x^3} - \frac{1}{(x+1)^3} - 3\left(\frac{1}{x^2} + \frac{1}{(x+1)^2}\right) + \frac{3\cdot4}{1\cdot2}\left(\frac{1}{x} - \frac{1}{x+1}\right)$$

$$\frac{1}{x^4(x+1)^4} = \frac{1}{x^4} + \frac{1}{(x+1)^4} - 4\left(\frac{1}{x^3} + \frac{1}{(x+1)^3}\right) + \frac{4\cdot5}{1\cdot2}\left(\frac{1}{x^2} + \frac{1}{(x+1)^2}\right) - \frac{4\cdot5\cdot6}{1\cdot2\cdot3}\left(\frac{1}{x} - \frac{1}{x+1}\right).$$

Cum iam ſummas praefixo ſigno $\int$ indicando ſit

$$\int\frac{1}{(x+1)^n} = \int\frac{1}{x^n} - 1, \quad \text{erit}$$

$$\int\frac{1}{x(x+1)} = 1,$$

$$\int\frac{1}{x^2(x+1)^2} = 2\int\frac{1}{x^2} - 1 - 2$$

$$\int\frac{1}{x^3(x+1)^3} = 1 - 3\left(2\int\frac{1}{x^2} - 1\right) + \frac{3\cdot4}{1\cdot2}$$

$$\int\frac{1}{x^4(x+1)^4} = 2\int\frac{1}{x^4} - 1 - 4 + \frac{4\cdot5}{1\cdot2}\left(2\int\frac{1}{x^2} - 1\right) - \frac{4\cdot5\cdot6}{1\cdot2\cdot3}.$$

12. In ſingulis his expreſſionibus numeros abſolutos commode in vnum colligere licet, et cum deinde ſit

$$\int\frac{1}{x^2} = \alpha\pi^2;\ \int\frac{1}{x^4} = \beta\pi^4;\ \int\frac{1}{x^6} = \gamma\pi^6;\ \int\frac{1}{x^8} = \delta\pi^2 \text{ etc.}$$

 habe-

habebimus

$$\int \frac{1}{x(x+1)} = 1$$
$$\int \frac{1}{x^2(x+1)^2} = 2\,\alpha\,\pi^2 - \frac{2\cdot 3}{1\cdot 2}$$
$$\int \frac{1}{x^3(x+1)^3} = -\,3\cdot 2\,\alpha\,\pi^2 + \frac{3\cdot 4\cdot 5}{1\cdot 2\cdot 3}$$
$$\int \frac{1}{x^4(x+1)^4} = 2\,\beta\,\pi^4 + \frac{4\cdot 5}{1\cdot 2}\,2\,\alpha\,\pi^2 - \frac{4\cdot 5\cdot 6\cdot 7}{1\cdot 2\cdot 3\cdot 4}$$
$$\int \frac{1}{x^5(x+1)^5} = -\,5\cdot 2\beta\pi^4 - \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3}\,2\,\alpha\,\pi^2 + \frac{5\cdot 6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5}$$
$$\int \frac{1}{x^6(x+1)^6} = 2\,\gamma\,\pi^6 + \frac{6\cdot 7}{1\cdot 2}\,2\,\beta\,\pi^4 + \frac{6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4}\,2\,\alpha\,\pi^2 - \frac{6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$$

etc.

vbi haec reductio notatu digna est obseruanda:

$$1 + \frac{n}{1} + \frac{n(n+1)}{1\cdot 2} + \ldots\ldots\ldots \frac{n(n+1)\ldots\ldots(2n-2)}{1\cdot 2\ldots\ldots(n-1)} = \frac{n(n+1)(n+2)\ldots(2n-1)}{1\cdot 2\cdot 3\ldots\ldots\ldots n}$$

fit enim summa ex lege nota

$$= \frac{(n+1)}{1}\frac{(n+2)}{2}\frac{(n+3)}{3}\ldots\ldots\frac{(2n-1)}{(n-1)}.$$

13. His igitur valoribus substitutis obtinebimus:

$$Z = \frac{1}{2^2}\cdot\frac{1}{2} + \frac{1}{2\cdot 2^4}(\alpha\pi^2 + 1) + \frac{1}{3\cdot 2^6}\left(\beta\pi^4 + \frac{2\cdot 3}{1\cdot 2} - 2\alpha\pi^2\right) + \frac{1}{4\cdot 2^8}\left(\gamma\pi^6 + \frac{3\cdot 4\cdot 5}{1\cdot 2\cdot 3} - \frac{3}{1}\cdot 2\alpha\pi^2\right)$$
$$+ \frac{1}{5\cdot 2^{10}}\left(\delta\pi^8 + \frac{4\cdot 5\cdot 6\cdot 7}{1\cdot 2\cdot 3\cdot 4} - \frac{4\cdot 5}{1\cdot 2}2\alpha\pi^2 - 2\beta\pi^4\right) + \frac{1}{6\cdot 2^{12}}\left(\varepsilon\pi^{10} + \frac{5\cdot 6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3}2\alpha\pi^2 - \frac{1}{5}\cdot 2\beta\pi^4\right)$$
$$+ \frac{1}{7\cdot 2^{14}}\left(\zeta\pi^{22} + \frac{6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} - \frac{6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4}2\alpha\pi^2 - \frac{6\cdot 7}{1\cdot 2}2\beta\pi^4 - 2\gamma\pi^6\right) + \text{etc.}$$

quae expressio in has series resoluitur:

$$Z = \frac{1}{2^2\cdot 2} + \frac{1}{2\cdot 2^4} - \frac{2\cdot 3}{2\cdot 3\cdot 2^6} + \frac{3\cdot 4\cdot 5}{2\cdot 3\cdot 4\cdot 2^8} + \frac{4\cdot 5\cdot 6\cdot 7}{2\cdot 3\cdot 4\cdot 5\cdot 2^{10}} + \frac{5\cdot 6\cdot 7\cdot 8\cdot 9}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 2^{12}} + \text{etc.}$$
$$+ \frac{\alpha\pi^2}{2\cdot 2^4} + \frac{\beta\pi^4}{3\cdot 2^6} + \frac{\gamma\pi^6}{4\cdot 2^8} + \frac{\delta\pi^8}{5\cdot 2^{10}} + \frac{\varepsilon\pi^{10}}{6\cdot 2^{12}} + \text{etc.}$$
$$- 2\alpha\pi^2\left(\frac{1}{3\cdot 2^6} + \frac{3}{1\cdot 4\cdot 2^8} + \frac{4\cdot 5}{1\cdot 2\cdot 5\cdot 2^{10}} + \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3\cdot 6\cdot 2^{12}} + \frac{6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 7\cdot 2^{14}}\right)$$
$$- 2\beta\pi^4\left(\frac{1}{5\cdot 2^{10}} + \frac{5}{1\cdot 6\cdot 2^{12}} + \frac{6\cdot 7}{1\cdot 2\cdot 7\cdot 2^{14}} + \frac{7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 6\cdot 2^{18}} + \frac{8\cdot 9\cdot 10\cdot 11}{1\cdot 2\cdot 3\cdot 4\cdot 9\cdot 2^{18}}\ \text{etc.}\right)$$
$$- 2\gamma\pi^6\left(\frac{1}{7\cdot 2^{14}} + \frac{7}{1\cdot 8\cdot 2^{16}} + \frac{8\cdot 9}{1\cdot 2\cdot 9\cdot 2^{18}} + \frac{9\cdot 10\cdot 11}{1\cdot 2\cdot 3\cdot 10\cdot 2^{20}} + \frac{10\cdot 11\cdot 12\cdot 13}{1\cdot 2\cdot 3\cdot 4\cdot 11\cdot 2^{22}} + \text{etc.}\right)$$

etc.

14.

14. Hinc deducimur ad iſtam ſeriem infinitam generalem, quae illas ſeries numericas omnes in ſe complectitur:

$$\frac{1}{n.\,2^{2n}}+\frac{n}{(n+1)\,2^{2n+2}}+\frac{(n+1)(n+2)}{2(n+2)\,2^{2n+4}}+\frac{(n+2)(n+3)(n+4)}{2.\;3.\;(n+3)\,2^{2n+6}}$$
$$+\frac{(n+3)(n+4)(n+5)(n+6)}{2.\;3.\;4\;(n+4)\,2^{2n+8}}+\text{etc.}$$

cuius igitur ſummam inueſtigari oportet. Quodſi enim huius ſeriei ſummam in genere hoc ſigno $S(n)$ indicemus, habebimus

$$Z=-\tfrac{1}{8}+S(1)+2\alpha\pi^2\left(\tfrac{1}{4.\,2^4}-S(3)\right)+2\beta\pi^4\left(\tfrac{1}{6.\,2^6}-S(5)\right)$$
$$+2\gamma\pi^6\left(\tfrac{1}{8.\,2^8}-S(7)\right)+2\delta\pi^8\left(\tfrac{1}{10.\,2^{10}}-S(9)\right)+\text{etc.}$$

Series autem noſtra generalis ita commodius exhiberi poteſt

$$n(n+1)\,S(n)=\frac{n+1}{2^{2n}}+\frac{nn}{2^{2n+2}}+\frac{n(n+1)(n+1)}{2.\;2^{2n+4}}+\frac{n(n+1)(n+2)(n+4)}{2.\;3.\;2^{2n+6}}$$
$$+\frac{n(n+1)(n+3)(n+5)(n+6)}{2.\;3.\;4.\;2^{2n+8}}+\frac{n(n+1)(n+4)(n+6)(n+7)(n+8)}{2.\;3.\;4.\;5.\;2^{2n+10}}+\text{etc.}$$

vbi denominatores numero n carent. Poſſunt etiam ſinguli termini ita per factores repraeſentari, vt ſtatuatur:

$$S(n)=A+AB+ABC+ABCD+ABCDE+\text{etc.}$$

eritque

$$A=\frac{1}{n.\,2^{2n}};\quad B=\frac{nn}{4(n+1)};\quad C=\frac{(n+1)(n+1)}{4.\;2\,n};\quad D=\frac{(n+2)(n+4)}{4.\,3\,(n+1)};$$
$$E=\frac{(n+3)(n+5)(n+6)}{4.\,4\,(n+2)(n+4)};\quad F=\frac{(n+4)(n+7)(n+8)}{4.\;5\,(n+3)(n+5)}\;\text{etc.}$$

vbi

vbi factor in genere hanc habet formam

$$\frac{(n+\lambda-1)(n+2\lambda-3)(n+2\lambda-2)}{4\lambda(n+\lambda-2)(n+\lambda)}.$$

15. Incipiamus a caſu ſimpliciſſimo $n=1$; et quia factor in genere eſt

$$=\frac{\lambda(2\lambda-2)(2\lambda-1)}{4\lambda(\lambda-1)(\lambda+1)}=\frac{2\lambda-1}{2\lambda+2}; \text{ erit}$$

$A=\frac{1}{4}$; $B=\frac{1}{8}$; $C=\frac{3}{6}$; $D=\frac{5}{8}$; $E=\frac{7}{10}$; $F=\frac{9}{12}$ etc.

vnde fit

$$S(1)=\tfrac{1}{4}+\tfrac{1}{4\cdot 8}\left(1+\tfrac{3}{6}+\tfrac{3\cdot 5}{6\cdot 7}+\tfrac{3\cdot 5\cdot 7}{6\cdot 8\cdot 10}+\tfrac{3\cdot 5\cdot 7\cdot 9}{6\cdot 8\cdot 10\cdot 12}+\text{etc.}\right).$$

Cum autem ſit

$$\sqrt{(1-1)}=1-\tfrac{1}{2}-\tfrac{1\cdot 1}{2\cdot 4}-\tfrac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}-\tfrac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}-\tfrac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}-\text{etc.}=0$$

erit $1+\frac{3}{6}+\frac{3\cdot 5}{6\cdot 8}+\frac{3\cdot 5\cdot 7}{6\cdot 8\cdot 10}+\text{etc.}=\frac{2\cdot 4}{1\cdot 1}\left(1-\frac{n}{1}\right)=4$

ideoque $S(1)=\frac{1}{4}+\frac{1}{8}=\frac{3}{8}$ et $-\frac{1}{8}+S(1)=\frac{1}{4}$.

16. Quo autem ſummas reliquarum ſerierum facilius definire queamus, loco $\frac{1}{2^2}$ ſcribamus x, vt ſit $x=\frac{1}{4}$, et cum habeamus:

$$S(n)=\tfrac{1}{n}x^n+\tfrac{n}{(n+1)}x^{n+1}+\tfrac{(n+1)(n+2)}{2(n+2)}x^{n+2}+\tfrac{(n+2)(n+3)(n+4)}{2\cdot 3(n+3)}x^{n+3}+\text{etc.}$$

quae caſu $n=1$ abit in

$$S(1)=x+\tfrac{1}{2}xx+\tfrac{3}{2\cdot 3}x^3+\tfrac{3\cdot 4\cdot 5}{2\cdot 3\cdot 4}x^4+\tfrac{4\cdot 5\cdot 6\cdot 7}{2\cdot 3\cdot 4\cdot 5}x^5+\text{etc.}$$

ſeu $S(1)=x+\frac{1}{2}xx+x^3+\frac{5}{2}x^4+\frac{6\cdot 7}{2\cdot 3}x^5+\frac{7\cdot 8\cdot 9}{2\cdot 3\cdot 4}x^6+\frac{8\cdot 9\cdot 10\cdot 11}{2\cdot 3\cdot 4\cdot 5}x^7+\text{etc.}$

vel $S(1)=x+\frac{1}{2}xx\left(1+\frac{2}{1}x+\frac{2\cdot 5}{1\cdot 2}xx+\frac{2\cdot 5\cdot 14}{1\cdot 2\cdot 5}x^3+\frac{2\cdot 5\cdot 14}{2\cdot 3\cdot 4\cdot 5}x^4+\text{etc.}\right)$

vel $S(1)=x+\frac{xx}{2}xx\left(1+\frac{3}{6}4x+\frac{3\cdot 5}{6\cdot 8}4^2x^2+\frac{3\cdot 5\cdot 7}{6\cdot 8\cdot 10}4^3x^3+\frac{3\cdot 5\cdot 7\cdot 9}{6\cdot 8\cdot 10\cdot 12}4^4x^4+\text{etc.}\right)$

At eſt

$$\sqrt{(1-4x)}=1-\tfrac{1}{2}4x-\tfrac{1\cdot 1}{2\cdot 4}4^2x^2-\tfrac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}4^3x^3-\tfrac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}4^4x^4-\text{etc.}$$

vnde $\frac{1\cdot 1}{2\cdot 4}4^2x^2\left(1+\frac{3}{6}\cdot 4x+\frac{3\cdot 5}{6\cdot 8}\cdot 4^2x^2+\text{etc.}\right)=1-2x-\sqrt{(1-4x)}$

ergo

ergo $S(1)=x+\frac{1-2x-\sqrt{(1-4x)}}{4}=\frac{1+2x-\sqrt{(1-4x)}}{4}$

sicque posito $x=\frac{1}{4}$ fit $S(1)=\frac{1}{4}(1+\frac{1}{2})=\frac{3}{8}$ vt supra.

17. Ponamus nunc $n=3$, sitque $S(3)=Q$, existente $S(1)=$

$$P=\frac{1+2x-\sqrt{(1-4x)}}{4}, \text{ ita vt sit}$$

$$P=x+\tfrac{1}{2}xx+\tfrac{2.3}{2.3}x^3+\tfrac{3.4.5}{2.3.4}x^4+\tfrac{4.5.6.7}{2.3.4.5}x^5+\text{etc.}$$

$$Q=\tfrac{1}{3}x^3+\tfrac{3}{4}x^4+\tfrac{4.5}{2.5}x^5+\tfrac{5.6.7}{2.3.6}x^6+\tfrac{6.7.8.9}{2.3.4.7}x^7+\text{etc.}$$

Ex his colligitur:

$$Pxx-Q=\tfrac{2}{3}x^3-\tfrac{1}{4}x^4-\tfrac{2.3.5}{2.3.5}x^5-\tfrac{3.4.5.8}{2.3.4.6}x^6-\tfrac{4.5.6.7.11}{2.3.4.5.7}x^7-\tfrac{5.6.7.8.9.14}{2.3.4.5.6.8}x^8-\text{etc.}$$

hincque differentiando

$$2Px+\frac{xxdP}{dx}-\frac{dQ}{dx}=2xx-x^3-\tfrac{2.3}{2.3}.5x^4-\tfrac{3.4.5}{2.3.4}.8x^5-\tfrac{4.5.6.7}{2.3.4.5}.11x^6-\tfrac{5.6.7.8.9}{2.3.4.5.6}.14x^7$$

at $\frac{xxdP}{dx}=xx+x^3+\tfrac{2.3}{2.3}.3x^4+\tfrac{3.4.5}{2.3.4}.4x^5+\tfrac{4.5.6.7}{2.3.4.5}.5x^6+\tfrac{5.6.7.8.9}{2.3.4.5.6}.6x^7$

cuius triplum ad priorem additum praebet

$$2Px+\frac{4xxdP}{dx}-\frac{dQ}{dx}=5xx+2x^3+4.\tfrac{2.3}{2.3}x^4+4.\tfrac{3.4.5}{2.3.4}x^5+4.\tfrac{4.5.6.7}{2.3.4.5}x^6+\text{etc.}$$

et $\quad 4Px \quad =4xx+2x^2+4.\tfrac{2.3}{2.3}x^4+4.\tfrac{3.4.5}{2.3.4}x^5+4.\tfrac{4.5.6.7}{2.3.4.5}x^6+\text{etc.}$

hinc $-2Px+\frac{4xxdP}{dx}-\frac{dQ}{dx}=xx$ et $dQ=4xxdP-2Pxdx-xxdx$

vnde ob $dP=\frac{1}{2}dx+\frac{dx}{2\sqrt{(1-4x)}}$ colligitur

$$dQ=-\tfrac{1}{2}xdx+\tfrac{1}{2}xdx\sqrt{(1-4x)}+\frac{2xxdx}{\sqrt{(1-4x)}}=-\tfrac{1}{2}xdx+\frac{xdx}{2\sqrt{(1-4x)}}$$

et integrando

$$Q=-\tfrac{1}{4}xx-\frac{(1+2x)}{24}\sqrt{(1-4x)}+\tfrac{1}{24}.$$

Ponatur $x=\frac{1}{4}$; quo casu fit $P=\frac{3}{8}$, erit $Q=\frac{5}{192}$ ita vt sit

$$S(1)=\tfrac{3}{8};\ S(3)=\tfrac{5}{192} \text{ et } -\tfrac{1}{8}+S(1)=\tfrac{1}{4};\ \tfrac{1}{4.24}-S(3)=-\tfrac{1}{96}.$$

18. Ponamus nunc in genere $S(n) = P$, et sequentem summam $S(n+2) = Q$ erit

$$P = \tfrac{1}{n}x^n + \tfrac{n}{n+1}x^{n+1} + \tfrac{n+1}{2}x^{n+2} + \tfrac{(n+2)(n+4)}{2 \cdot 3}x^{n+3} + \tfrac{(n+3)(n+5)(n+6)}{2 \cdot 3 \cdot 4}x^{n+4} \text{ etc.}$$

$$Q = \tfrac{1}{n+2}x^{n+2} + \tfrac{n+2}{n+3}x^{n+3} + \tfrac{n+3}{2}x^{n+4} + \tfrac{(n+4)(n+6)}{2 \cdot 3}x^{n+5} + \tfrac{(n+5)(n+7)(n+8)}{2 \cdot 3 \cdot 4}x^{n+6}$$

ex quibus colligitur fore

$$Q = Px - \tfrac{1}{2}(n+1)\int P\,dx - \tfrac{1}{2}(n-1)xx\int \tfrac{P\,dx}{xx}.$$

Hinc ergo ex valore $S(n)$ definiri potest valor $S(n+2)$ excepto quidem casu $n = 1$, quia tum in $\int \frac{P\,dx}{xx}$ occurret $\int \frac{dx}{x}$.

At quia iam constat casus

$$S(3) = \tfrac{1 - 6xx - (1+2x)\sqrt{(1-4x)}}{24}$$

si hic pro P sumatur, fiet

$$S(5) = Px - 2\int P\,dx - xx\int \tfrac{P\,dx}{xx},$$

quae per integrationem euoluta dat

$$S(5) = \tfrac{1}{60}(1 - 15xx + 10x^3 - (1 + 2x - 9xx)\sqrt{(1-4x)}).$$

Posito ergo $x = \frac{1}{4}$ fit $S(5) = \frac{7}{32 \cdot 60} = \frac{7}{1920}$, ideoque

$$\tfrac{1}{6 \cdot 2^6} - S(5) = -\tfrac{1}{960}.$$

19. Sit porro $n = 5$ et

$$P = \tfrac{1}{60}(1 - 15xx + 10x^3 - (1 + 2x - 9xx)\sqrt{(1-4x)})$$

erit $S(7) = Px - 3\int P\,dx - 2xx\int \frac{P\,dx}{xx}$),

quibus integralibus euolutis tandem reperitur:

S (7)

$S(7) = \frac{1}{112}(1 - 28xx + 56x^3 - 14x^4 - (1 + 2x - 22xx + 20x^3)\sqrt{(1-4x)})$

Posito ergo $x = \frac{1}{4}$ fit

$$S(7) = \frac{1}{112}\left(1 - \frac{7}{4} + \frac{7}{8} - \frac{7}{128}\right) = \frac{9}{2^{11} \cdot 7}$$

hincque $\frac{1}{8 \cdot 2^8} - S(7) = - \frac{1}{2^{10} \cdot 7}$.

Quare si, quae hactenus inuenimus, colligamus, etiam sequentes valores coniectura facile obtinebimus:

$$S(1) - \frac{1}{2 \cdot 2^2} = \frac{1}{4} = \frac{1}{1 \cdot 2 \cdot 2^1}$$

$$S(3) - \frac{1}{4 \cdot 2^4} = \frac{1}{96} = \frac{1}{3 \cdot 4 \cdot 2^3}$$

$$S(5) - \frac{1}{6 \cdot 2^6} = \frac{1}{960} = \frac{1}{5 \cdot 6 \cdot 2^5}$$

$$S(7) - \frac{1}{8 \cdot 2^8} = \frac{1}{2^{10} \cdot 7} = \frac{1}{7 \cdot 8 \cdot 2^7}.$$

20. Hinc igitur tandem pro Z sequentem valorem adipiscimur:

$$Z = \frac{1}{4} - \frac{\alpha \pi^2}{3 \cdot 4 \cdot 2^2} - \frac{\beta \pi^4}{5 \cdot 6 \cdot 2^4} - \frac{\gamma \pi^6}{7 \cdot 8 \cdot 2^6} - \frac{\delta \pi^8}{9 \cdot 10 \cdot 2^8} - \frac{\varepsilon \pi^{10}}{11 \cdot 12 \cdot 2^{10}} - \text{etc.}$$

ita vt sit

$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \frac{1}{2}\pi\pi Z$, seu

$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \frac{1}{8}\pi\pi - \frac{2\alpha\pi^4}{3 \cdot 4 \cdot 2^4} - \frac{2\beta\pi^6}{5 \cdot 6 \cdot 2^4} - \frac{2\gamma\pi^8}{7 \cdot 8 \cdot 2^8} - \text{etc.}$

Cuius seriei vt summam inuestigemus, consideremus π tanquam quantitatem variabilem, factoque $\frac{\pi}{2} = \Phi$ ponamus

$$\frac{\alpha \Phi^4}{3 \cdot 4} + \frac{\beta \Phi^6}{5 \cdot 6} + \frac{\gamma \Phi^8}{7 \cdot 8} + \frac{\delta \Phi^{10}}{9 \cdot 10} + \text{etc.} = s$$

erit $\frac{dds}{d\Phi^2} = \alpha\Phi^2 + \beta\Phi^4 + \gamma\Phi^6 + \delta\Phi^8 + \text{etc.} = z$

vnde formemus:

$$2zz = 2\alpha\alpha\Phi^4 + 4\alpha\beta\Phi^6 + 4\alpha\gamma\Phi^8 + 4\alpha\delta\Phi^{10} + \text{etc.}$$
$$\qquad\qquad\qquad\qquad + 2\beta\beta \qquad + 4\beta\gamma.$$

 Iam

Iam quia $\beta = \frac{2\alpha\alpha}{5}$; $\gamma = \frac{4\alpha\beta}{7}$; $\delta = \frac{4\alpha\gamma + 2\beta\beta}{9}$ etc. erit

$$2\int zz\,d\Phi = \beta\Phi^5 + \gamma\Phi^7 + \delta\Phi^9 + \text{etc.} = z\Phi - \alpha\Phi^3$$

hincque

$$2zz\,d\Phi = z\,d\Phi + \Phi\,dz - 3\alpha\Phi\Phi\,d\Phi.$$

21. Cum nunc sit $\alpha = \frac{1}{6}$, reperitur per integrationem

$$z = \frac{1}{2} - \frac{\Phi}{2\,\text{tang.}\,\Phi}, \text{ vti tentanti patebit.}$$

Hinc cum $dds = z\,d\Phi^2$, colligitur

$$\frac{ds}{d\Phi} = \int z\,d\Phi = \frac{1}{2}\Phi - \frac{1}{2}\int\frac{\Phi\,d\Phi}{\text{tang.}\,\Phi}$$

et $s = \frac{1}{4}\Phi^2 - \frac{1}{2}\int d\Phi\int\frac{\Phi\,d\Phi}{\text{tang.}\,\Phi}$, ideoque

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \frac{1}{8}\pi\pi - \frac{1}{2}\Phi\Phi + \int d\Phi\int\frac{\Phi\,d\Phi}{\text{tang.}\,\Phi}$$

et ob $\Phi = \frac{\pi}{2}$ seu $\pi = 2\Phi$ erit

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \int d\Phi\int\frac{\Phi\,d\Phi}{\text{tang.}\,\Phi} = \frac{\pi}{2}\int\frac{\Phi\,d\Phi}{\text{tang.}\,\Phi} - \int\frac{\Phi\Phi\,d\Phi}{\text{tang.}\,\Phi}$$
$$= 2\int\Phi\,d\Phi\,l\,\text{sin.}\,\Phi - \frac{\pi}{2}\int d\Phi\,l\,\text{sin.}\,\Phi$$

at est $\int d\Phi\,l\,\text{sin.}\,\Phi = -\frac{\pi\,l\,2}{2}$

ergo

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} = \frac{\pi\pi}{4}\,l\,2 + 2\int\Phi\,d\Phi\,l\,\text{sin.}\,\Phi$$

vbi integralibus ita sumtis, vt euanescant, posito $\Phi = 0$, statui deinceps debet $\Phi = \frac{\pi}{2}$, vt obtineatur summa seriei propositae. Etsi autem integratio institui nequit, tamen per quadraturas eius valor

valor definiri poteſt. At vero ipſa ſeries ante inventa per Z maxime eſt idonea ad ſummam proxime determinandam.

22. Hinc anſam arripio accuratius iſtam ſeriem perpendendi:

$$P = \frac{1}{n} x^n + \frac{n}{n+1} x^{n+1} + \frac{(n+1)}{2} x^{n+2} + \frac{(n+2)(n+4)}{2 \cdot 3} x^{n+3} + \frac{(n+3)(n+5)(n+6)}{2 \cdot 3 \cdot 4} x^{n+4} + \text{etc.}$$

cuius valores pro caſibus, quibus n eſt numerus integer impar, methodo ſingulari determinauimus, qui ita ſe habent:

ſi $n = 1$; $P = \frac{1}{4}(1 + 2x - \sqrt{(1 - 4x)})$

$n = 3$; $P = \frac{1}{24}(1 - 6xx - (1 + 2x)\sqrt{(1 - 4x)})$

$n = 5$; $P = \frac{1}{60}(1 - 15xx + 10x^3 - (1 + 2x - 9xx)\sqrt{(1 - 4x)})$

$n = 7$; $P = \frac{1}{112}(1 - 28xx + 56x^3 - 14x^4 - (1 + 2x - 22xx + 20x^3)\sqrt{(1 - 4x)})$.

Quare cum ſummatio in genere ad aequationem differentialem reduci queat, operae pretium videtur examinare, quomodo his caſibus iſti valores ſatisfaciant. Conueniet autem potius differentialia conſiderari, quae ſunt

ſi $n = 1$; $\frac{dP}{dx} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{(1 - 4x)}}\right)$

$n = 3$; $\frac{dP}{dx} = \frac{1}{2}\left(-x + \frac{x}{\sqrt{(1 + 4x)}}\right)$

$n = 5$; $\frac{dP}{dx} = \frac{1}{2}\left(-x + xx + \frac{x - 3xx}{\sqrt{(1 - 4x)}}\right)$

$n = 7$; $\frac{dP}{dx} = \frac{1}{2}\left(-x + 3xx - x^3 + \frac{x - 5xx + 5x^3}{\sqrt{(1 - 4x)}}\right.$.

23. In genere autem eſt differentiando:

$$\frac{dP}{dx} = x^{n-1} + nx^n + \frac{(n+1)(n+2)}{1 \cdot 2} x^{n+1} + \frac{(n+2)(n+3)(n+4)}{1 \cdot 2 \cdot 3} x^{n+2} + \text{etc.}$$

Ponamus $x = yy$ et $\frac{dP}{dx} = \frac{dP}{2y\,dy} = s$, vt habeamus

$$s = y^{2n-2} + ny^{2n} + \frac{(n+1)(n+2)}{1.\ 2} y^{2n+2} + \frac{(n+2)(n+3)(n+4)}{1.\ 2.\ 3} y^{2n+4} + \text{etc.}$$

vnde fit

$$y^{2-n}s = y^n + ny^{n+2} + \frac{(n+1)(n+2)}{1.\ 2} y^{n+4} + \frac{(n+2)(n+3)(n+4)}{1.\ 2.\ 3} y^{n+6}$$

hincque porro:

$$\frac{dd.y^{2-n}s}{dy^2} = n(n-1)y^{n-2} + \frac{n(n+1)(n+2)}{1} y^n + \frac{(n+1)(n+2)(n+3)(n+4)}{1.\ 2} y^{n+2} + \text{etc.}$$

Est vero etiam:

$$\frac{1}{2\,dy} d.\frac{s}{yy} = (n-2)y^{2n-5} + \frac{n(n+1)(n+2)}{1.\ 2} y^{2n-1} + \frac{(n+1)(n+2)(n+3)(n+4)}{1.\ 2.\ 3} y^{2n+1}$$

quae multiplicata per y^{5-2n} et denuo differentiata producit

$$\frac{1}{4\,dy^2} d.\left(y^{5-2n} d.\frac{s}{yy}\right) = n(n-1)y + \frac{n(n+1)(n+2)}{1} y^3 + \frac{(n+1)(n+2)(n+3)(n+4)}{1.\ 2} y^5 + \text{etc.}$$

quae series per superiorem etiam est

$$= \frac{y^{3-n}\,dd(y^{2-n}s)}{dy^2},$$

ita vt inter s et y hanc habeamus aequationem

$$d\left(y^{5-2n} d.\frac{s}{yy}\right) = 4y^{3-n}\,dd.(y^{2-n}s).$$

24. Sumto elemento dy constante haec aequatio euoluta dat:

$$y^{3-2n}dds+(1-2n)y^{2-2n}dyds-4(1-n)y^{1-2n}sdy^2=$$
$$4y^{5-2n}dds+8(2-n)y^{4-2n}dyds+4(2-n)(1-n)y^{3-2n}sdy^2$$

quae per y^{2n-1} multiplicata abit in hanc:

$$\left.\begin{array}{l} yy(1-4yy)dds+(1-2n)ydyds-4(1-n)sdy^2 \\ \quad -8(2-n)y^2dyds-4(2-n)(1-n)yysdy^2 \end{array}\right\}=0$$

quae posito $yy=x$ sumtoque dx constante transformatur in hanc

$$\left.\begin{array}{l} xx(1-4x)dds+(1-n)xdxds-(1-n)sdx^2 \\ \quad -2(5-2n)xxdxds-(2-n)(1-n)sxdx^2 \end{array}\right\}=0$$

vbi est $s=\frac{dP}{dx}$ seu $P=\int sdx$. Integrationes autem hac lege institui debent, vt existente x infinite paruo fiat,

$$\frac{ds}{dx}=(n-1)x^{n-2};\ s=x^{n-1} \quad \text{et } P=\tfrac{1}{n}x^n.$$

25. Si haec aequatio per seriem infinitam integretur, incipiendo a termino x^{n-1}, ipsa proposita series reproducitur, sed etiam initium fieri potest a termino constante x^0, vnde quoque integrale obtinetur, quod autem nostris conditionibus non satis facit; verum praeterea aliud integrale elici potest, quod cum illo coniunctim negotium conficit. Fingatur ergo

$$s=0+Ax+Bx^2+Cx^3+Dx^4+Ex^5+Fx^6+\text{etc.}$$

erit $\frac{ds}{dx}=A+2Bx+3Cxx+4Dx^3+5Ex^4+6Fx^5+$ etc.

et $\frac{dds}{dx^2}=2B+6Cx+12Dxx+20Ex^4+38Fx^4+42Gx^5+$etc.

quibus

quibus ſeriebus ſubſtitutis fieri oportet:

$$\left.\begin{array}{llllll} -(1-n)O & -(2-n)(1-n)Ox & -(2-n)(1-n)Ax^2 & -(2-n)(1-n)Bx^3 & -(2-n)(1-n)Cx^4 & -(2-n)(1-n)Dx^5 \\ & +(1-n)A & +2(1-n)B & +3(1-n)C & +4(1-n)D & +5(1-n)E \\ & -(1-n)A & -2(5-2n)A & -4(5-2n)B & -6(5-2n)C & -8(5-2n)D \;\text{etc.} \\ & & -(1-n)B & +6C & -(1-n)D & -(1-n)E \\ & & +2B & -8B & +12D & +20E \\ & & & & +24C & -48D \end{array}\right\} = 0$$

quae aequatio reducitur ad hanc formam:

$$\left.\begin{array}{l} -(1-n)O-(2-n)(1-n)Ox+(3-n)Bx^2+2(4-n)Cx^3+3(5-n)Dx^4+4(6-n)Ex^5 \\ \qquad\qquad -(3-n)(4-n)A-(5-n)(6-n)B-(7-n)(8-n)C-(9-n)(10-n)D \end{array}\right\}\text{etc.} = 0.$$

26. Singulis ergo terminis ad nihilum perductis, fieri debet $O=0$ niſi ſit $n=1$, at pro reliquis coefficientibus habebitur

$$B=\frac{(3-n)(4-n)}{1(3-n)}A=\frac{4-n}{1}A$$

$$C=\frac{(5-n)(5-n)}{2(4-n)}B=\frac{(4-n)(6-n)}{1.\,2}A$$

$$D=\frac{(7-n)(8-n)}{3(5-n)}C=\frac{(6-n)(7-n)(8-n)}{1.\,2.\,3}A$$

$$E=\frac{(9-n)(10-n)}{4(6-n)}D=\frac{(7-n)(8-n)(9-n)(10-n)}{1.\,2.\,3,\,4}A$$

etc.

vnde lex progreſſionis eſt manifeſta. Caſu autem $n=1$ quantitas O manet indefinita, tum autem aequationi ſatisfit, reliquis coefficientibus omnibus annihilatis, ita vt ſit $s=O$, etiamſi ex his determinationibus valores quoque finiti pro iis aſſumi poſſent veluti

$$B=\tfrac{3}{1}A;\; C=\tfrac{4.5}{1.2}A;\; D=\tfrac{5.6.7}{1.2.3}A \text{ etc.}$$

vnde integrale completum foret:

$$s=O+A+(x+\tfrac{3}{1}x^2+\tfrac{4.5}{1.2}x^3+\tfrac{5.6.7}{1.2.3}x^4+\tfrac{6.7.8.9}{1.2.3.4}x^5+\text{etc.}$$

27. Simili modo pro reliquis caſibus, quibus n eſt numerus integer, fit quidem $O=0$, ſed A numerus arbitrarius, ſed praeterea alius quidam coeffi-

coefficiens etiam non definitur, quem propterea pro lubitu assumere licet. Quare si is $=0$ ponatur, habebitur integrale terminis finitis contentum, quod ita se habebit:

si $n=3$; $O=0$; A indef. $B=0$; $C=0$ etc.
si $n=4$; $O=0$; A indef. $B=a$; $C=0$ etc.
si $n=5$; $O=0$; A indef. $B=-A$; $C=0$; $D=0$ etc.
si $n=6$; $O=0$; A indef. $B=-2A$; $C=0$; $D=0$ etc.
si $n=7$; $O=0$; A indef. $B=-3A$; $C=A$; $D=0$; etc.
si $n=8$; $O=0$; A indef. $B=-4A$; $C=\frac{2\cdot 3}{1\cdot 2}A$; $D=0$; $E=0$ etc.
si $n=9$; $O=0$; A indef. $B=-5A$; $C=\frac{3\cdot 4}{1\cdot 2}A$; $D=-\frac{1\cdot 2\cdot 3}{1\cdot 2\cdot 3}A$; $E=0$
si $n=10$; $O=0$; A indef. $B=-6A$; $C=\frac{4\cdot 5}{1\cdot 2}A$; $D=-\frac{2\cdot 3\cdot 4}{1\cdot 2\cdot 3}A$; $E=0$
si $n=11$; $O=0$; A indef. $B=-7A$; $C=\frac{5\cdot 6}{1\cdot 2}A$; $D=-\frac{3\cdot 4\cdot 5}{1\cdot 2\cdot 3}A$; $E=\frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 2\cdot 3\cdot 4}A$.

28. Ecce ergo pro omnibus casibus, quibus n est numerus integer positiuus praeter $n=2$ integralia particularia, vnde partes rationales formularum supra pro $\frac{dp}{dx}$ inuentarum colligi possunt:

si $n=1$; $s=O$
si $n=3$; $s=Ax$
si $n=4$; $s=Ax$
si $n=5$; $s=A(x-xx)$
si $n=6$; $s=A(x-2xx)$
si $n=7$; $s=A(x-3xx+x^3)$
si $n=8$; $s=A(x-4xx+3x^3)$
si $n=9$; $s=A(x-5xx+6x^3-x^4)$
si $n=10$; $s=A(x-6xx+10x^3-4x^4)$
si $n=11$; $s=A(x-7xx+15x^3-10x^4+x^5)$
si $n=12$; $s=A(x-8xx+21x^3-20x^4+5x^6)$.

29. Pro numero ergo quocunque n integrale hoc particulare eſt

$$s = A\left\{x - \frac{(n-4)}{1}xx + \frac{(n-5)(n-6)}{1.\ 2}x^3 - \frac{(n-6)(n-7)(n-8)}{1.\ 2.\ 3}x^4 + \frac{(n-7)(n-8)(n-9)(n-10)}{1.\ 2.\ 3.\ 4}x^5 - \text{etc.}\right\}$$

quae ſeries etſi in infinitum continuata ſatisfacit, tamen cum quispiam terminus euanuerit, ſequentes omnes omittere licet, quippe qui ſeorſim ſumti aliud integrale particulare praeberent.

Ceterum hinc euidens eſt, quamlibet harum formularum ex binis praecedentibus ita definiri, vt ſi pro caſibus $n = \nu$, $n = \nu + 1$, $n = \nu + 2$, valor ipſius s ponantur s, s', s'' futurum ſit

$$s'' = s' - sx$$

ſiquidem conſtans A in omnibus eundem valorem retineat. Atque vi huius legis pro caſu $n = 2$ ſtatui oportet $s = 0$. Verum vti iam monui, haec integralia particularia noſtris conditionibus non ſatisfaciunt, verumtamen partes rationales ſuppeditant vti mox videbimus.

30. Vt autem completa integralia eruamus, alia integralia particularia inueſtigemus, quae partes irrationales praebeant. In hunc finem ponamus

$$s = \frac{t}{\sqrt{(1-4x)}} \text{ erit } ds = \frac{dt}{\sqrt{(1-4x)}} + \frac{2t\,dx}{(1-4x)^{\frac{3}{2}}}$$

$$\text{et}\quad dds = \frac{ddt}{\sqrt{(1-4x)}} + \frac{4dxdt}{(1-4x)^{\frac{3}{2}}} + \frac{12t\,dx^2}{(1-4x)^{\frac{5}{2}}}$$

quibus

quibus ſubſtitutis noſtra aequatio differentio - differentialis abibit in hanc formam:

$$\left.\begin{array}{l} xx(1-4x)ddt-(n-1)x\,dt\,dx+(n-1)t\,dx^2 \\ \qquad -2(2n-3)xx\,dt\,dx-n(n-1)tx\,dx^2 \end{array}\right\} = 0.$$

Ponatur hic

$$t = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 + \text{etc.}$$

factaque ſubſtitutione peruenitur ad hanc aequationem:

$$\begin{array}{l} 0=(n-1)A-n(n-1)A-(n-3)Cxx-2(n-4)Dx^3-3(n-5)Ex^4 \\ \qquad\qquad -(n-2)(n-3)B-(n-4)(n-5)C-(n-6)(n-7)D \end{array} \text{ etc.}$$

niſi ergo $n = 1$ debet eſſe $A = 0$, et pro reliquis fit

$$C = -\frac{(n-2)(n-3)}{1\,(n-3)}B = -\frac{(n-2)}{1}B$$

$$D = -\frac{(n-4)(n-5)}{2\,(n-4)}C = +\frac{(n-2)(n-5)}{1\cdot 2}B$$

$$E = -\frac{(n-6)(n-7)}{3\,(n-5)}D = -\frac{(n-2)(n-6)(n-7)}{1\cdot 2\cdot 3}B.$$

31. Ex his ergo valores finiti ipſius t pro ſingulis numeris integris n ita ſe habebunt:

ſi $n = 1$; $t = A$
ſi $n = 2$; $t = Bx$
ſi $n = 3$; $t = Bx$
ſi $n = 4$; $t = B(x-2xx)$
ſi $n = 5$; $t = B(x-3xx)$
ſi $n = 6$; $t = B(x-4xx+2x^3)$
ſi $n = 7$; $t = B(x-5xx+5x^3)$
ſi $n = 8$; $t = B(x-6xx+9x^3-2x^4)$
ſi $n = 9$; $t = B(x-7xx+14x^3-7x^4)$
ſi $n = 10$; $t = B(x-8xx+20x^3-16x^4+2x^5)$

et in genere

$$t = B \left\{ x - \frac{(n-2)}{1} xx + \frac{(n-2)(n-5)}{1 \cdot 2} x^3 - \frac{(n-2)(n-6)(n-7)}{1 \cdot 2 \cdot 3} x^4 + \frac{(n-2)(n-7)(n-8)(n-9)}{1 \cdot 2 \cdot 3 \cdot 4} x^5 - \text{etc} \right\}$$

vbi iterum vt ante $t'' = t' - t\,x$.

32. Quemadmodum fupra feriem P defignauimus per $S(n)$, fi feriem $s = \frac{dP}{dx}$ per $\Sigma(n)$ defignemus, erit generatim

$\Sigma(1) = -A + \frac{B}{\sqrt{(1-4x)}}$ hic $A = -\frac{1}{2}$ et $B = \frac{1}{2}$

$\Sigma(2) = 0 + \frac{2Bx}{\sqrt{(1-4x)}}$ vt hae formae ad feriem pro-

$\Sigma(3) = Ax + \frac{Bx}{\sqrt{(1-4x)}}$ pofitam accommodentur.

$\Sigma(4) = Ax + \frac{B(x-2xx)}{\sqrt{(1-4x)}}$

$\Sigma(5) = A(x-xx) + \frac{B(x-3xx)}{\sqrt{(1-4x)}}$

$\Sigma(6) = A(x-2xx) + \frac{B(x-4xx+2x^3)}{\sqrt{(1-4x)}}$

$\Sigma(7) = A(x-3xx+x^3) + \frac{B(x-5xx+5x^3)}{\sqrt{(1-4x)}}$

$\Sigma(8) = A(x-4xx+3x^3) + \frac{B(x-6xx+9x^3-2x^4)}{\sqrt{(1-4x)}}$

$\Sigma(9) = A(x-5xx+6x^3-x^4) + \frac{B(x-7xx+14x^3-7x^4)}{\sqrt{(1-4x)}}$

$\Sigma(10) = A(x-6xx+10x^3-4x^4) + \frac{B(x-8xx+20x^3-16x^4+2x^5)}{\sqrt{(1-4x)}}$

etc.

33. Quoniam hi valores feriem recurrentem conftituunt, cum quisque aequetur praecedentium vltimo, demto penultimo per x multiplicato, terminus generalis feu valor $\Sigma(n)$ finite exprimi poterit, erit enim ex proprietate ferierum recurrentium:

$$\Sigma(n) = M \frac{(1+\sqrt{(1-4x)})^n}{2} + N \frac{(1-\sqrt{(1-4x)})^n}{2}$$

vbi

vbi ex binis primis coefficientes M et N ita definiuntur vt ſit

$$M = \frac{(A+B)(1-2x-\sqrt{(1-4x)})}{2x\sqrt{(1-4x)}} = \frac{A+B}{x\sqrt{(1-4x)}}\left(\frac{1-\sqrt{(1-4x)}}{2}\right)^2$$

$$N = \frac{(B-A)(1-2x+\sqrt{(1-4x)})}{2x\sqrt{(1-4x)}} = \frac{B-A}{x\sqrt{(1-4x)}}\left(\frac{1+\sqrt{(1-4x)}}{2}\right)^2$$

Cum autem pro noſtro caſu ſit $A = -\frac{1}{2}$ et $B = \frac{1}{2}$ erit

$$\Sigma(n) = \frac{1}{x\sqrt{(1-4x)}}\left(\frac{1+\sqrt{(1-4x)}}{2}\right)^2\left(\frac{1-\sqrt{(1-4x)}}{2}\right)^n$$

$$\text{ſeu } \Sigma(n) = \frac{x}{\sqrt{(1-4x)}}\left(\frac{1-\sqrt{(1-4x)}}{2}\right)^{n-2} = \frac{dP}{dx}$$

exiſtente

$$P = \frac{1}{n}x^n + \frac{n}{n+1}x^{n+1} + \frac{(n+1)}{2}x^{n+2} + \frac{(n+2)(n+4)}{2\cdot 3}x^{n+3} + \frac{(n+3)(n+5)(n+6)}{2\cdot 3\cdot 4}x^{n+4} + \text{etc.}$$

34. Huius ergo ſeriei valor, quem ponimus P eſt

$$P = \int \frac{x\,dx}{\sqrt{(1-4x)}}\left(\frac{1-\sqrt{(1-4x)}}{2}\right)^{n-2}$$

ad quod integrale inueniendum ponatur $\frac{1-\sqrt{(1-4x)}}{2} = y$ erit

$$dy = \frac{dx}{\sqrt{(1-4x)}} \text{ et } x = y - yy, \text{ vnde fit}$$

$$P = \int dy\,(y-yy)\,y^{n-2} = \frac{y^n}{n} - \frac{y^{n+1}}{n+1} \text{ ideoque}$$

$$P = \frac{n+1-ny}{n(n+1)}\cdot y^n \text{ ſeu}$$

$$P = \frac{n+2+n\sqrt{(1-4x)}}{2n(n+1)}\left(\frac{1-\sqrt{(1-4x)}}{2}\right)^n = S(n) \text{ poſito } x = \frac{1}{4}$$

vnde pro formulis ſupra (14) expoſitis colligitur

$$S(n) = \frac{n+2}{2^{n+1}\,n(n+1)}$$

hincque vti formulae ibi ſe habent:

$$\frac{1}{(n+1)2^{n+1}} - S(n) = -\frac{1}{2^n(n+1)n}$$

quae expreſſio prorſus congruit cum iis, quas ſupra §. 19. ſola inductione nixi dedimus, ita vt nunc quidem nullum amplius dubium ſupereſſe poſſit.

35. Deinde memorabile eſt, huius aequationis differentio-differentialis

$$\left.\begin{array}{l} xx(1-4x)dds-(n-1)xdxds+(n-1)sdx^2 \\ \quad +2(2n-5)xxdxds-(n-1)(n-2)sxdx^2 \end{array}\right\} = 0$$

integrale completum et quidem algebraicum aſſignari poſſe, quod ex praecedentibus ita ſe habet:

$$s = \frac{Cx}{\sqrt{(1-4x)}}\left(\frac{1+\sqrt{(1-4x)}}{2}\right)^{n-2} + \frac{Dx}{\sqrt{(1-4x)}}\left(\frac{1-\sqrt{(1-4x)}}{2}\right)^{n-2}$$

quod quomodo per integrationem inde erui queat, non tam facile patet. Hinc tamen ſtatim intelligitur, ſubſtitutionem $s = \frac{xu}{\sqrt{(1-4x)}}$ plurimum eſſe profuturam, poſito enim in §. 30; $t = ux$ haec aequatio naſcitur

$$\begin{array}{l} xx(1-4x)ddu-(n-3)xdxdu-(n-2)(n-3)xudx^2 = 0 \text{ ſeu} \\ \quad +2(2n-7)xxdxdu \end{array}$$

$$\begin{array}{l} x(1-4x)ddu-(n-3)dxdu-(n-2)(n-3)udx^2 = 0 \\ \quad +2(2n-7)xdxdu \end{array}$$

cuius ergo integrale eſt

$$u = C\left[\frac{1+\sqrt{(1-4x)}}{2}\right]^{n-2} + D\left[\frac{1-\sqrt{(1-4x)}}{2}\right]^{n-2}.$$

36.

36. Si in hac aequatione ponatur $\sqrt{(1-4x)} = y$, et elementum dy vt conſtans introducatur, prodibit iſta aequatio ſimplicior

$$(1-yy)ddu + 2(n-3)ydydu - (n-2)(n-3)udy^2 = 0$$

cuius integrale iam eſſe conſtat:

$$u = C\left(\frac{1+y}{2}\right)^{n-2} + D\left(\frac{1-y}{2}\right)^{n-2}.$$

Quo igitur pateat, quomodo haec inde elici queat, ponamus $n = m + 2$ vt habeamus:

$$(1-yy)ddu + 2(m-1)ydydu - m(m-1)udy^2 = 0$$

vbi commode tentari poſſe patet hanc poſitionem

$$u = (\alpha + \beta y)^m$$

vnde fit

$$du = m\beta dy(\alpha+\beta y)^{m-1} \text{ et } ddu = m(m-1)\beta\beta dy^2(\alpha+\beta y)^{m-2}$$

quo facto erit

$$m(m-1)(\alpha+\beta y)^{m-2}(\beta\beta(1-yy)+2\beta(\alpha y+\beta yy)-\alpha\alpha-2\alpha\beta y-\beta\beta yy) = 0$$

ideoque

$$\beta\beta = \alpha\alpha; \text{ ergo } u = C(1 \pm y)^m.$$

Atque ob ſignum ambiguum obtinebitur integrale completum:

$$u = C(1+y)^m + D(1-y)^m.$$

37. Ceterum notaſſe iuuabit hanc poſtremam aequationem

$$(1-yy)ddu + 2(m-1)ydydu - m(m-1)udy^2 = 0$$

reddi integrabilem, ſi diuidatur per $(1 \pm y)^m$. Prior vero aequatio:

$$x(1-4x)ddu - (n-3)dxdu - (n-2)(n-3)udx^2 = 0$$
$$+ 2(2n-7)xdxdu$$

in-

integrabilis euadet si multiplicetur per

$$x^{-n+3}\,du - \frac{(n-2)}{2}\,x^{-n+2}\,u\,dx.$$

In genere autem proposita hac aequatione

$$\left.\begin{aligned} xx(A+Bx)ddu+\tfrac{1}{2}(2\alpha+\lambda)\,A\,x\,dx\,du+\tfrac{1}{2}\alpha(\lambda-2)A\,u\,dx^2 \\ +\tfrac{1}{2}(2\alpha+\lambda+1)B\,xx\,dx\,du+\tfrac{1}{2}\alpha(\lambda-1)\,B\,x\,u\,dx^2 \end{aligned}\right\}=0$$

si ea multiplicetur per

$$x^{\lambda-2}\,du+\alpha\,x^{\lambda-3}\,u\,dx,$$

fiet integrabilis, eritque integrale:

$$\tfrac{1}{2}x^{\lambda}(A+Bx)\,du^2+\alpha x^{\lambda-1}(A+Bx)\,u\,du\,dx+\tfrac{1}{2}\alpha\alpha x^{\lambda-1}(A+Bx)\,u^2\,dx^2=\tfrac{1}{2}C\,dx^2$$

seu $x^{\lambda}du^2+2\,\alpha x^{\lambda-1}u\,du\,dx+\alpha\alpha x^{\lambda-2}u^2dx^2=\frac{C\,dx^2}{A+Bx}$

ergo $x^{\frac{1}{2}\lambda}du+\alpha x^{\frac{1}{2}\lambda-1}\,u\,dx=\frac{dx\sqrt{C}}{\sqrt{(A+Bx)}}$ hincque

$$u=x^{-\alpha}\int x^{\alpha-\frac{1}{2}\lambda}\frac{dx\sqrt{C}}{\sqrt{(A+Bx)}}.$$

38. In hac autem aequatione generali nostra superior non continetur: quare conditiones aequationis huius

$$xx(A+Bx)\,ddu+x(C+Dx)\,du\,dx+(E+Fx)\,u\,dx^2=0$$

accuratius inuestigemus, vt per

$$x^{\lambda-2}\,du+\alpha\,x^{\lambda-3}\,u\,dx$$

multiplicata fiat integrabilis. Ac primo quidem integrale fit

$$\tfrac{1}{2}x^{\lambda}(A+Bx)\,du^2+\alpha x^{\lambda-1}(A+Bx)\,u\,du\,dx+\frac{\alpha E}{\lambda-2}x^{\lambda-2}\,uu\,dx^2 + \frac{\alpha F}{\lambda-1}u^2\,dx^2=G\,dx^2$$

requiritur autem, vt sit primo

$$C=(\alpha+\tfrac{1}{2}\lambda)\,A;\quad D=(\alpha+\tfrac{1}{2}\lambda+\tfrac{1}{2})\,B$$

tum

tum vero triplici modo

I. vel $E = \frac{1}{2}\alpha(\lambda-2)A$ et $F = \frac{1}{2}\alpha(\lambda-1)B$ qui est casus superior

II. vel $\lambda = 2\alpha+2$; $F = \frac{1}{2}\alpha(2\alpha+1)B$ manente E indefinito

III. vel $\lambda = 2\alpha+1$; $E = \frac{1}{2}\alpha(2\alpha-1)A$ manente F indefinito.

39. En ergo duas aequationes differentio-differentiales satis late patentes, quas hac methodo integrale licet:

$$\text{I. } xx(A+Bx)ddu + (2\alpha+1)Axdxdu + Eudx^2 + (2\alpha+\tfrac{5}{2})Bxxdxdu + \tfrac{1}{2}\alpha(2\alpha+1)Bxudx^2 = 0$$

quae per

$$x^{2\alpha}du + \alpha x^{2\alpha-1}udx$$

multiplicata integrale dat

$$\tfrac{1}{2}x^{2\alpha+1}(A+Bx)du^2 + \alpha x^{2\alpha+1}(A+Bx)ududx + \tfrac{1}{2}Ex^{2\alpha}uudx^2 + \tfrac{1}{2}\alpha\alpha Bx^{2\alpha+1}uudx^2 = Gdx^2.$$

Altera vero forma est

$$\text{II. } xx(A+Bx)ddu + (2\alpha+\tfrac{1}{2})Axdxdu + \tfrac{1}{2}\alpha(2\alpha-1)Audx^2 + (2\alpha+1)Bxxdxdu + Fxudx^2 = 0$$

quae per

$$x^{2\alpha-1}du + \alpha x^{2\alpha-2}udx$$

multiplicata istud suppeditat integrale:

$$\tfrac{1}{2}x^{2\alpha+1}(A+Bx)du^2 + \alpha x^{2\alpha}(A+Bx)ududx + \tfrac{1}{2}\alpha\alpha Ax^{2\alpha-1}u^2dx^2 + \tfrac{1}{2}Fx^{2\alpha}u^2dx^2 = Gdx^2.$$

In priori si ponatur $A = 1$, $B = -4$ et $2\alpha+1 = -n+3$, atque $E = 0$ prodit aequatio in §. 35 proposita.

40. Verum alia datur via ſummam progreſſionis §. 23

$$\frac{dP}{dx} = x^{n-1} + \frac{n}{1}x^n + \frac{(n+1)(n+2)}{1.\ 2}x^{n+1} + \frac{(n+2)(n+3)(n+4)}{1.\ 2.\ 3}x^{n+2} + \text{etc.}$$

inueſtigandi, in qua quia x vt conſtans ſpectatur, conſideremus hanc ſeriem

$$s = 1 + \frac{n}{2}a^2 + \frac{(n+1)(n+2)}{2.\ 4}a^4 + \frac{(n+2)(n+3)(n+4)}{2.\ 4.\ 6}a^6 + \text{etc.}$$

vbi $aa = 2x$ et $\frac{dP}{dx} = x^{n-1}s$. Iam in ſubſidium vocetur haec ſeries

$$\frac{(1+ay)^{-n+1} + (1-ay)^{-n+1}}{2} = 1 + \frac{(n-1)n}{1.\ 2}aay^2$$

$$+ \frac{(n-1)n(n+1)(n+2)}{1.\ 2.\ 3.\ 4}a^4y^4 + \text{etc.}$$

pro vna ſcribamus breuitatis gratia

$$1 + Aa^2y^2 + Ba^4y^4 + Ca^6y^6 + \text{etc.}$$

eritque

$$s = 1 + \frac{1}{n-1}Aa^2 + \frac{1.\ 3}{(n-1)n}Ba^4 + \frac{1.\ 3.\ 5}{(n-1)(n+1)}Ca^6 \text{etc.}$$

Statuatur nunc

$$s = \frac{1}{2}\int dz(1 + Aa^2y^2 + Ba^4y^4 + Ca^6y^6 + \text{etc.})$$

ac fieri oportet

$$\int yy\,dz = \frac{1}{n-1}\int dz$$

$$\int y^4\,dz = \frac{3}{n}\int yy\,dz$$

$$\int y^6\,dz = \frac{5}{n+1}\int y^4\,dz$$

ideoque ingenere

$$\int y^{2\lambda}\,dz = \frac{2\lambda-1}{n+\lambda-2}\int y^{2\lambda-2}\,dz$$

si post integrationem ipsi y certus valor tribuatur.

41. Ponamus ergo esse ingenere:

$$\int y^{2\lambda} dz = \frac{2\lambda - 1}{n+\lambda-2} \int y^{2\lambda-2} dz + \frac{Q y^{2\lambda-1}}{n+\lambda-2}$$

hincque differentiando et per $y^{2\lambda-2}$ diuidendo colligitur

$$(n+\lambda-2) yy dz = (2\lambda - 1) dz + y dQ + (2\lambda-1) Q dy$$

quae aequatio pro omnibus numeris λ locum habere debet, vnde erit

tam $yy dz = 2 dz + 2 Q dy$

quam $(n-2) yy dz = -dz + y dQ - Q dy$

ergo $dz = \frac{2 Q dy}{yy - 2} = \frac{y dQ - Q dy}{(n-2) yy + 1}$ vnde fit

$$\frac{dQ}{Q} = \frac{-(2n-3) y dy}{2 - yy}, \text{ et } Q = (2 - yy)^{n - \frac{3}{2}}$$

hincque

$$dz = -2 dy (2 - yy)^{n - \frac{5}{2}}.$$

Quare posito post integrationem

$y = \sqrt{2}$ fit $\int y^{2\lambda} dz = \frac{2\lambda - 1}{n+\lambda-2} \int y^{2\lambda-2} dz$

reperiturque

$$s = \frac{\int dy (2 - yy)^{n - \frac{5}{2}} ((1 + ay)^{-n+1} + (1 - ay)^{-n+1})}{2 \int dy (2 - yy)^{n - \frac{5}{2}}}$$

si post integrationem ponatur $y = \sqrt{2}$.

42. Etsi autem haec methodus statim pro summa quaesita formulam integralem exhibet, tamen

 verum

verum valorem in expreſſione algebraica non oſtendit. In ſuperioribus autem vidimus eſſe

$$\frac{dP}{dx} = \frac{x}{\sqrt{(1-4x)}} \left(\frac{1-\sqrt{(1-4x)}}{2}\right)^{n-2},$$

vnde concludimus fore hic

$$s = \frac{x^{2-n}}{\sqrt{(1-4x)}} \left(\frac{1-\sqrt{(1-4x)}}{2}\right)^{n-2} = \frac{1}{\sqrt{(1-4x)}} \left(\frac{1-\sqrt{(1-4x)}}{2x}\right)^{n-2}.$$

Quare ſi ſtatuamus $2x = aa$, et ſuperioris formulae integralis caſu $y = \sqrt{2}$ erit valor algebraicus:

$$s = \frac{1}{\sqrt{(1-2aa)}} \left(\frac{1-\sqrt{(1-2aa)}}{aa}\right)^{n-2}$$

quae circumſtantia minime contemnenda videtur, cum forte hinc plura alia praeclara in hoc genere deriuare liceat.

DIGRES-